viernes, 24 de junio de 2022

Cálculo Integral

$$L = \int_a^b \sqrt{1 + \left(\frac{d}{dx}(y(x))\right)^2}\; dx$$

$$\left.\begin{array}{l} x = x(t) \\ y = y(t) \end{array}\right\} \Rightarrow \begin{array}{l} a = x(t_1) \\ b = x(t_2) \end{array} \quad , \quad \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \quad , \quad dx = \frac{dx(t)}{dt} \cdot dt$$

$$L = \int_{t_1}^{t_2} \sqrt{1 + \left[\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}\right]^2} \cdot x'(t) \cdot dt$$

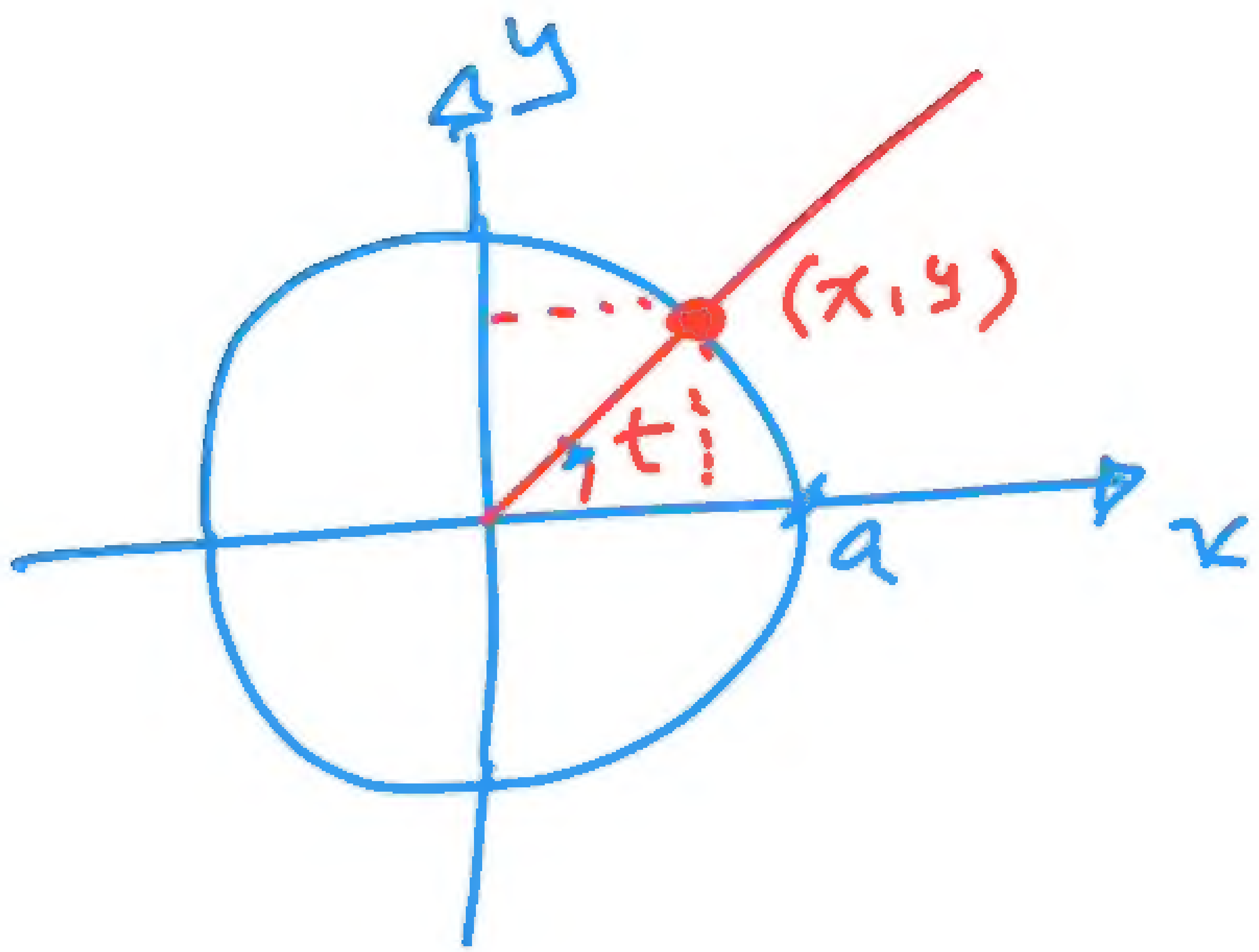
$$L = \int_{t_1}^{t_2} \sqrt{\frac{(x'(t))^2 + (y'(t))^2}{(x'(t))^2}} \cdot x'(t)\, dt$$

$$L = \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2} \cdot \frac{1}{|x'(t)|} \cdot x'(t) . dt$$

$$\boxed{L = \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2} . dt \quad , \quad \text{con } x'(t) > 0 \text{ en } [t_1, t_2]}$$

Ejemplo:

$$x = a\cos t$$
$$y = a\,\mathrm{sen}\,t$$



$$L = 4\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Si $x = 0 \Rightarrow \cos t = 0 \Rightarrow t = \pi/2$

Si $x = a \Rightarrow \cos t = 1 \Rightarrow t = 0$

$$L = -4\int_{\pi/2}^{0} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\cdot dt$$

$$x(t) = a\cos t$$
$$x'(t) = -a\,\mathrm{sen}\,t < 0 \quad \forall t \in [0, \pi/2]$$

$$L = 4\int_0^{\pi/2} \sqrt{a^2\mathrm{sen}^2 t + a^2\cos^2 t}\,dt = 4a\int_0^{\pi/2} dt = 4a\left(t\Big/_0^{\pi/2}\right. = 2a\pi$$

Ejercicio de la cicloide pag 283 de Maynard Kong.

$$x(t) = a(t - \operatorname{sen} t) \Rightarrow x'(t) = a(1 - \cos t) > 0, \forall t \in]0, 2\pi] ; \quad a > 0$$

$$y(t) = a(1 - \cos t) \Rightarrow y'(t) = a(\operatorname{sen} t)$$

$$L = \int_0^{2\pi} \sqrt{a^2(1-\cos t)^2 + a^2 \operatorname{sen}^2 t}\; dt =$$

Se tiene :

$$\sqrt{a^2\left(1 + \cos^2 t - 2\cos t + \operatorname{sen}^2 t\right)} = \sqrt{a^2(2 - 2\cos t)} = \sqrt{2a^2(1-\cos t)}$$

$$= \sqrt{2}\, a \sqrt{1 - \cos t} = 2a \sqrt{\frac{1-\cos t}{2}} =$$

$$= 2a\sqrt{\operatorname{Sen}^2\left(\frac{t}{2}\right)} = 2a\left|\operatorname{Sen}\left(\frac{t}{2}\right)\right|$$

$$t \in [0, 2\pi] \Rightarrow \frac{t}{2} \in [0, \pi] \Rightarrow \left|\operatorname{Sen}\left(\frac{t}{2}\right)\right| = \operatorname{Sen}\left(\frac{t}{2}\right)$$

$\therefore$

$$L = 2a\int_0^{2\pi} \operatorname{sen}\left(\frac{t}{2}\right) dt = -4a \cdot \cos\left(\frac{t}{2}\right)\Big|_0^{2\pi}$$

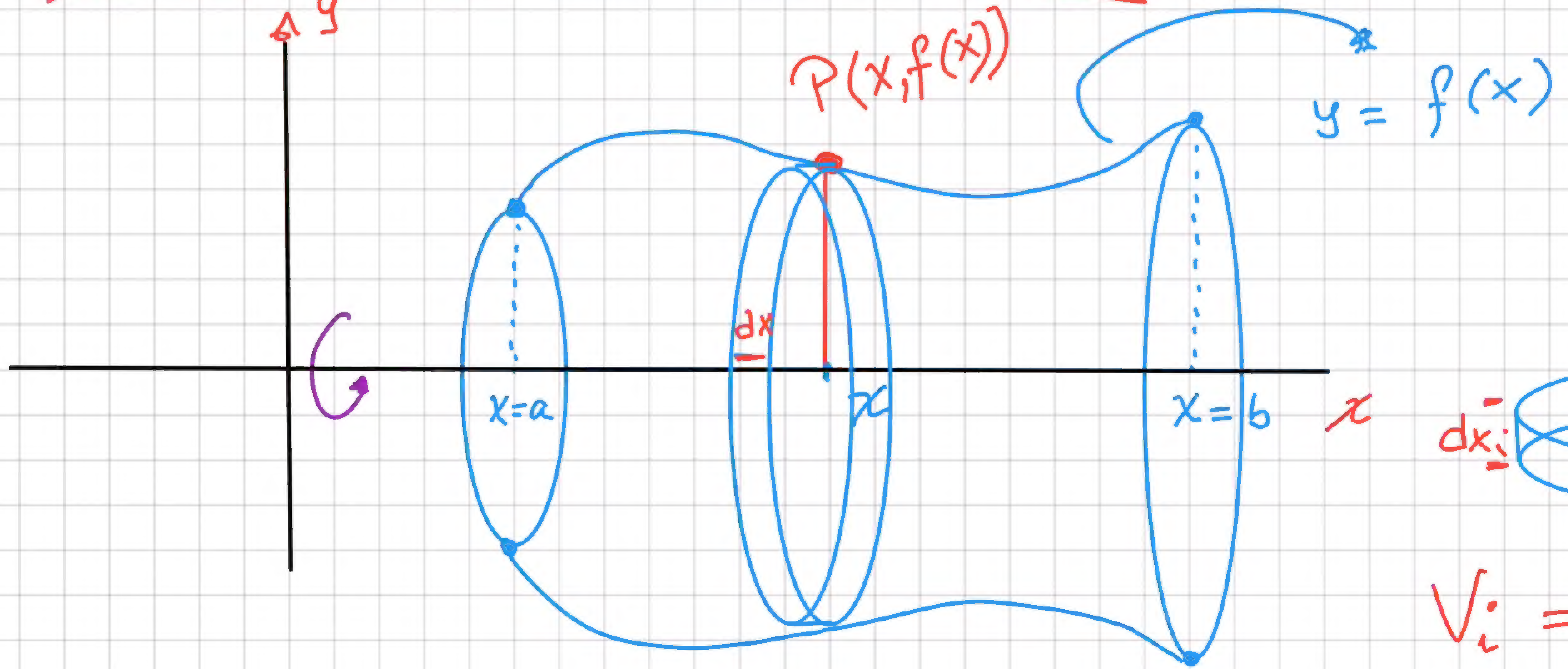
$$= -4a\left[\cos(\pi) - \cos(0)\right]$$

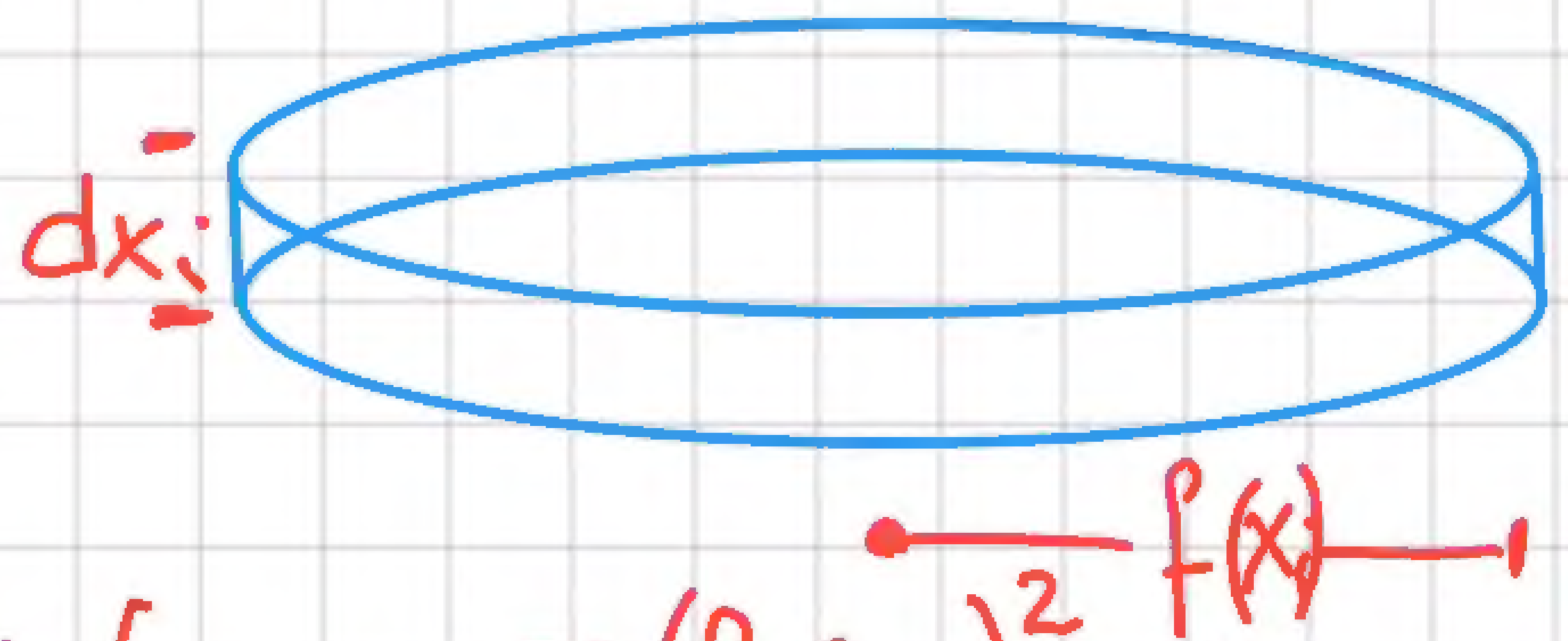
$$= -4a\left[-1-1\right] = 8a \text{ unidades}$$

Volúmenes de sólidos

Ver pág 293 del libro Cálc. Int. de Maynard Kong.

Volúmenes de sólidos de revolución





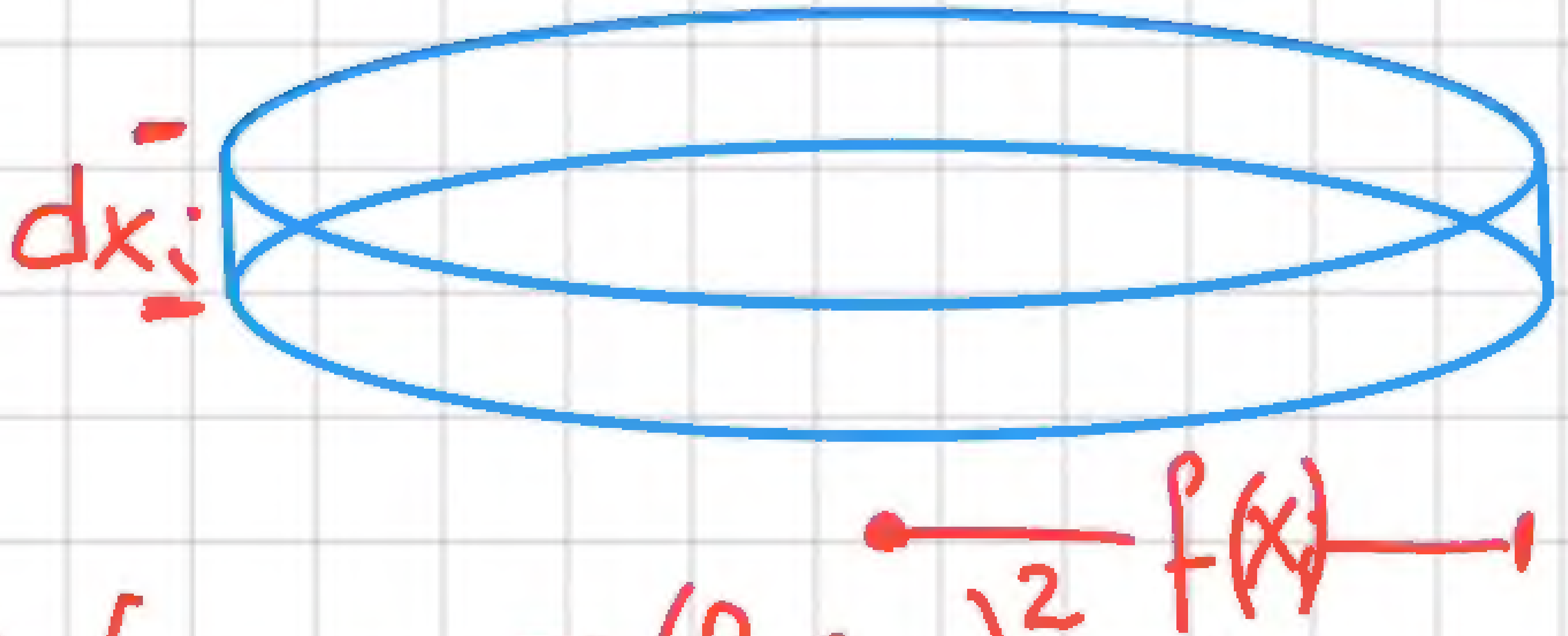
$$V_i = \pi (f(x_i))^2 . dx_i$$

V = volumen del sólido de revolución

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} V_i$$

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} \pi \left(f(x_i)\right)^2 \cdot \Delta x_i$$

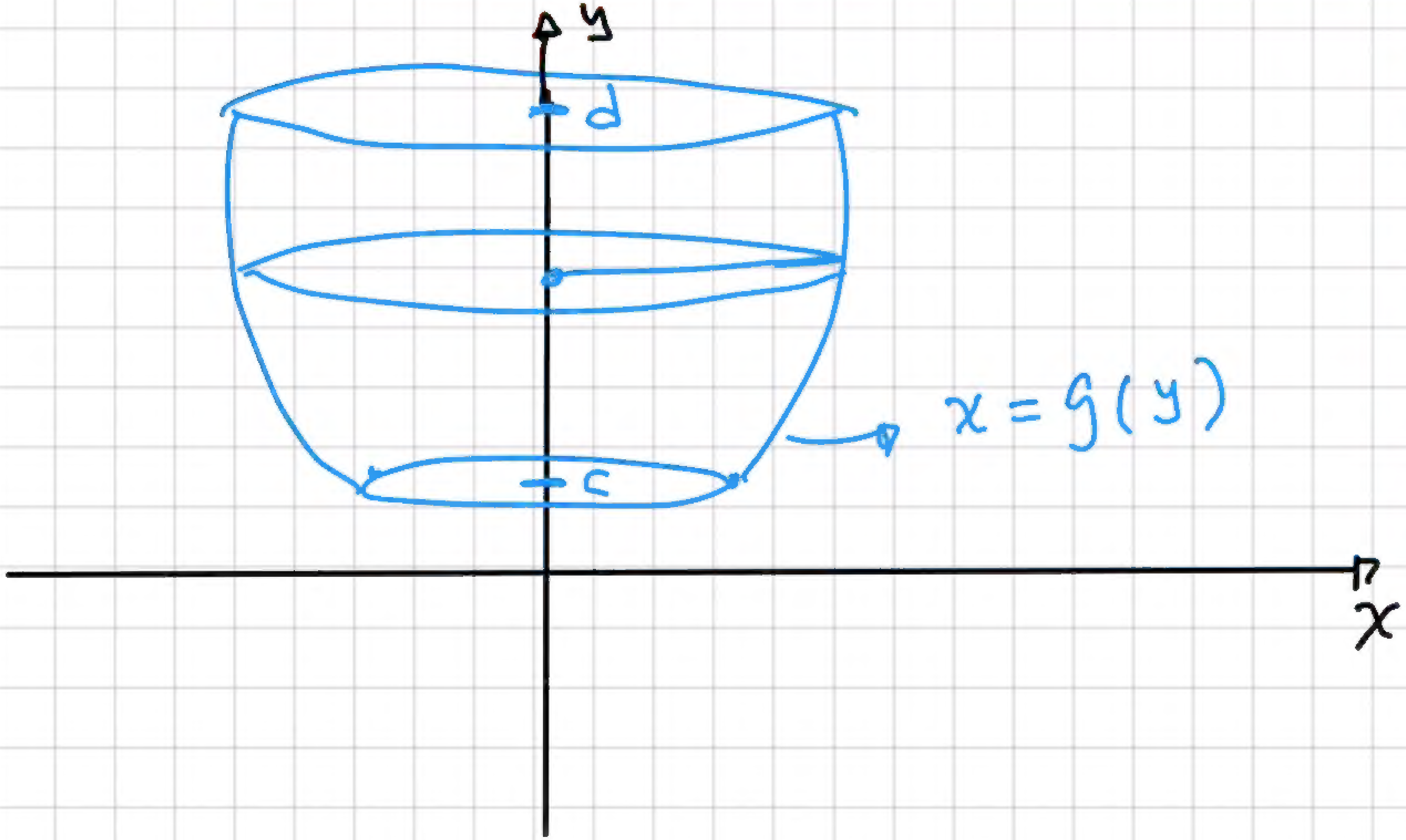
$$V = \pi \underbrace{\lim_{n \to \infty} \sum_{i=1}^{n} \left(f(x_i)\right)^2 \cdot \Delta x_i}$$



$$V = \pi \int_a^b [f(x)]^2 \, dx$$

$$V_i = \pi \left(f(x_i)\right)^2 \cdot dx_i$$

$$V = \pi \int_a^b [f(x)]^2 \, dx \; u^3$$



$$V = \pi \int_c^d (g(y))^2 \, dy \quad u^3$$